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Conformally flat pencils of metrics, Frobenius structures and a modified Saito construction

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Abstract

The structure of a Frobenius manifold encodes the geometry associated with a flat pencil of metrics. However, as shown in the authors' earlier work [L. David, I.A.B. Strachan, Compatible metrics on manifolds and non-local bi-Hamiltoninan structures, Int. Math. Res. Notices 66 (2004) 3533–3557], much of the structure comes from the compatibility property of the pencil rather than from the flatness of the pencil itself. In this paper conformally flat pencils of metrics are studied and examples, based on a modification of the Saito construction, are developed.

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1. Introduction

The Saito construction [10] of a flat structure on the orbit space $\mathbb{C}^n/W$, where W is a Coxeter group, has played a foundational role in many areas of mathematics. It is a central construction in singularity theory and contains the kernel of the definition of a Frobenius manifold, this having been done many years before the introduction of a Frobenius manifold by Dubrovin [2].

The initial motivation for this paper was the observation that one may repeat the Saito construction starting with a metric of constant non-zero sectional curvature s. One easily obtains a

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pencil of metrics $(h, \tilde{h})$ on the orbit space $\mathbb{C}^n/W$ (when $s > 0$) or $\mathbb{H}^n \otimes \mathbb{C}/W$ (when $s < 0$) of a Coxeter group W. The pencil $(h, \tilde{h})$ has interesting geometric properties: it is conformally related to the flat pencil provided by the classical Saito construction, the metric h has constant sectional curvature s and, as it turns out, the metric $\tilde{h}$ is flat. This modified Saito construction is developed in Section 2. The construction may be applied, locally, to any Frobenius manifold and this is also illustrated in Section 2.

A flat pencil of metrics leads, via the results of Dubrovin and Novikov [3] and Magri [7], to bi-Hamiltonian structures and the theory of integrable systems. The flatness of such pencils is required for the locality of the bi-Hamiltonian structures; however, one may introduce curvature – resulting in non-local Hamiltonian operators – in such a way as to preserve the bi-Hamiltonian property. Geometrically one requires a compatible pencil of metrics rather than a flat pencil.

In the authors' earlier work [1] the geometry of compatible metrics was studied in detail—this generalizing the results of Dubrovin [4] from flat pencils of metrics to compatible (and curved) pencils of metrics. In Section 3 we continue this study. One way to construct examples is to scale a known flat pencil of metrics by a conformal factor. This introduces curvature but the new metrics remain compatible. The geometry of such conformally scaled compatible pencils is studied in Section 4 and provides a general scheme into which the modified Saito construction of Section 2 falls.

The rest of this section outlines some standard notations and earlier results.

1.1. Compatible metrics on manifolds

Let M be a smooth manifold. We shall use the following notations: $\mathcal{X}(M)$ for the space of smooth vector fields on M; $\mathcal{E}^1(M)$ for the space of smooth 1-forms on M. For a pseudo-Riemannian metric g on M, ∇^g will denote its Levi–Civita connection and R^g its curvature. The metric g induces an inverse metric g^* on the cotangent bundle T^*M of M, i.e. a bilinear form g^*_p on every cotangent space T^*_pM which varies smoothly with respect to p. The 1-form corresponding to $X \in \mathcal{X}(M)$ via the pseudo-Riemannian duality defined by g will be denoted $g(X)$. Conversely, if $\alpha \in \mathcal{E}^1(M)$, the corresponding vector field will be denoted $g^*(\alpha)$.

Following [1] we recall the basic theory of compatible metrics on manifolds; the flat case has been treated in [4]. Let $(g, \tilde{g})$ be an arbitrary pair of metrics on M. Recall that the pair $(g, \tilde{g})$ defines a multiplication [4]:

$$\alpha \circ \beta := \nabla^g_{g^*(\alpha)}\beta - \nabla^{\tilde{g}}_{g^*(\alpha)}\beta \quad \forall \alpha, \beta \in \mathcal{E}^1(M) \tag{1}$$

on T^*M (or on TM, by identifying TM with T^*M using the metric $\tilde{g}$). For every constant λ we define the inverse metric $g^*_\lambda := g^* + \lambda \tilde{g}^*$, which, we will assume, will always be non-degenerate and whose Levi–Civita connection and curvature tensor will be denoted ∇^λ and R^λ, respectively. The metrics g and $\tilde{g}$ are almost compatible [9] if, by definition, the relation

$$g^*_\lambda(\nabla^\lambda_X \alpha) = g^*(\nabla^g_X \alpha) + \lambda \tilde{g}^*(\nabla^{\tilde{g}}_X \alpha) \tag{2}$$

holds, for every $X \in \mathcal{X}(M)$, $\alpha \in \mathcal{E}^1(M)$ and constant λ. The almost compatibility condition is equivalent with the vanishing of the integrability tensor N_K of $K := g^*\tilde{g} \in \mathrm{End}(TM)$, defined by the formula:

$$N_K(X, Y) = -[KX, KY] + K[KX, Y] + K[X, KY] - K^2[X, Y] \quad \forall X, Y \in \mathcal{X}(M)$$

and implies the following two relations:

$$g^*(\nabla^{\tilde{g}}_{\tilde{g}^*(\gamma)}\alpha - \nabla^{g}_{\tilde{g}^*(\gamma)}\alpha) = \tilde{g}^*(\nabla^{\tilde{g}}_{g^*(\gamma)}\alpha - \nabla^{g}_{g^*(\gamma)}\alpha) \quad \forall \alpha, \gamma \in \mathcal{E}^1(M) \tag{3}$$

and

$$\tilde{g}^*(\alpha \circ \beta, \gamma) = \tilde{g}^*(\alpha, \gamma \circ \beta) \quad \forall \alpha, \beta, \gamma \in \mathcal{E}^1(M). \tag{4}$$

Recall now that two almost compatible metrics $(g, \tilde{g})$ are compatible [9] if, by definition, the relation:

$$g^*_\lambda(R^\lambda_{X,Y}\alpha) = g^*(R^g_{X,Y}\alpha) + \lambda\tilde{g}^*(R^{\tilde{g}}_{X,Y}\alpha)$$

holds, for every $\alpha \in \mathcal{E}^1(M)$, $X, Y \in \mathcal{X}(M)$ and constant λ. The compatibility condition has several alternative formulations: if the metrics $(g, \tilde{g})$ are almost compatible, then they are compatible if and only if the relation

$$g^*(\nabla^{\tilde{g}}_X\alpha - \nabla^g_X\alpha, \nabla^{\tilde{g}}_Y\beta - \nabla^g_Y\beta) = g^*(\nabla^{\tilde{g}}_Y\alpha - \nabla^g_Y\alpha, \nabla^{\tilde{g}}_X\beta - \nabla^g_X\beta) \tag{5}$$

holds for every $X, Y \in \mathcal{X}(M)$ and $\alpha, \beta \in \mathcal{E}^1(M)$, or, in terms of the multiplication "$\circ$" associated to the pair $(g, \tilde{g})$:

$$(\alpha \circ \beta) \circ \gamma = (\alpha \circ \gamma) \circ \beta \quad \forall \alpha, \beta, \gamma \in \mathcal{E}^1(M). \tag{6}$$

If the metrics $(g, \tilde{g})$ are compatible and $R^\lambda = 0$ for all λ then $(g, \tilde{g})$ are said to form a flat pencil of metrics [4].

1.2. The Dubrovin correspondence

We end this section by recalling the Dubrovin correspondence [4] between flat pencils of metrics and Frobenius manifolds and its generalizations [1]. We first recall the definition of a Frobenius manifold.

Definition 1 (Dubrovin [4]). M is a Frobenius manifold if a structure of a Frobenius algebra (i.e. a commutative, associative algebra with multiplication denoted by "$\bullet$", an identity element "e" and an inner product "$\langle , \rangle$" satisfying the invariance condition $\langle a \bullet b, c\rangle = \langle a, b \bullet c\rangle$) is specified on the tangent space T_pM at any point $p \in M$ smoothly depending on the point p, such that

(i) The invariant metric $\tilde{g} = \langle , \rangle$ is a flat metric on M.
(ii) The identity vector field e is covariantly constant with respect to the Levi–Civita connection $\nabla^{\tilde{g}}$ of the metric $\tilde{g}$:

$$\nabla^{\tilde{g}} e = 0.$$

(iii) The (4, 0)-tensor $\nabla^{\tilde{g}}(\bullet)$ defined by the formula:

$$\nabla^{\tilde{g}}(\bullet)(X, Y, Z, V) := \tilde{g}(\nabla^{\tilde{g}}_X(\bullet)(Y, Z), V) \quad \forall X, Y, Z \in \mathcal{X}(M)$$

is symmetric in all arguments.
(iv) A vector field E - the Euler vector field—must be determined on M such that

$$\nabla^{\tilde{g}}(\nabla^{\tilde{g}} E) = 0$$

and that the corresponding one-parameter group of diffeomorphisms acts by conformal transformations of the metric and by rescalings of the multiplication "$\bullet$".

Using the flat coordinates $\{t^i\}$ of the metric $\langle , \rangle$ one may express the multiplication in terms of the derivatives of a scalar prepotential F:

$$\left\langle \frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j} \bullet \frac{\partial}{\partial t^k} \right\rangle = \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k},$$

where the t^1-dependence of F is fixed by the condition

$$\left\langle \frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j} \right\rangle = \frac{\partial^3 F}{\partial t^1 \partial t^i \partial t^j}.$$

The associativity condition then becomes an overdetermined partial differential equation for the prepotential F known as the Witten–Dijkgraaf–Verlinde–Verlinde equation.

Recall now that if $(M, \bullet, \tilde{g}, E)$ is a Frobenius manifold then we can define an inverse metric g^* by the relation $g^*\tilde{g} = E\bullet$. The metrics $(g, \tilde{g})$ form a flat pencil on the open subset of M where $E\bullet$ is a tangent bundle automorphism, satisfying some additional conditions (the quasi-homogeneity conditions). Conversely, a (regular) quasi-homogeneous flat pencil of metrics on a manifold determines a Frobenius structure on that manifold. This construction is known in the literature as the Dubrovin correspondence [4].

It turns out that the key role in the Dubrovin correspondence is played not by the flatness property of the metrics but rather by their compatibility. Weaker versions of the Dubrovin correspondence have been developed in [1]. Following [1] we recall now a weak version of the Dubrovin correspondence. In general, a pair of metrics $(g, \tilde{g})$ together with a vector field E on a manifold M such that the endomorphism $T(u) := g(E) \circ u$ of T^*M is an automorphism (such a pair of metrics is called "regular") determines a multiplication $u \bullet v := u \circ T^{-1}(v)$ on T^*M (or on TM, by identifying TM with T^*M using the metric $\tilde{g}$). If the metrics $(g, \tilde{g})$ are compatible, then the multiplication "$\bullet$" is associative, commutative, with identity $g(E)$ on T^*M, the metrics g, $\tilde{g}$ are "$\bullet$"-invariant and $g^*\tilde{g} = E\bullet$. Moreover, if E satisfies the relations

$$L_E(\tilde{g}) = D\tilde{g}, \qquad \nabla^g_X(E) = \frac{1-d}{2} X \quad \forall X \in \mathcal{X}(M), \tag{7}$$

for some constants D and d, then $(M, \bullet, \tilde{g}, E)$ is a weak $\mathcal{F}$-manifold, i.e. the following conditions are satisfied:

(1) The metric $\tilde{g}$ and the multiplication "$\bullet$" define a Frobenius algebra at every tangent space of M.
(2) The vector field E – the Euler vector field – rescales the metric $\tilde{g}$ and the multiplication "$\bullet$" by constants and has an inverse E^{-1} with respect to the multiplication "$\bullet$", which is a smooth vector field on M.
(3) The $(4, 0)$-tensor field $\nabla^{\tilde{g}}(\bullet)$ of M satisfies the symmetries:

$$\nabla^{\tilde{g}}(\bullet)(E, Y, Z, V) = \nabla^{\tilde{g}}(\bullet)(Y, E, Z, V) \quad \forall Y, Z, V \in \mathcal{X}(M). \tag{8}$$

Conversely, a weak $\mathcal{F}$-manifold $(M, \bullet, \tilde{g}, E)$ determines a pair $(g, \tilde{g})$ of compatible metrics, with g defined by the formula $g^*\tilde{g} = E\bullet$, and the Euler vector field E satisfies relations (7). Therefore, there is a one to one correspondence between (regular) compatible pencils of metrics $(g, \tilde{g})$ with a vector field E satisfying relations (7) and weak $\mathcal{F}$-manifolds.

Under a certain curvature condition on the metrics $(g, \tilde{g})$ – see Theorem 23 of [1] – the tensor $\nabla^{\tilde{g}}(\bullet)$ is symmetric in all arguments and then $(M, \bullet, \tilde{g}, E)$ is called an $\mathcal{F}$-manifold. Note that in this case $(M, \bullet)$ is an F-manifold [6,8], i.e. the relation

$$L_{X\bullet Y}(\bullet) = X \bullet L_Y(\bullet) + Y \bullet L_X(\bullet) \quad \forall X, Y \in \mathcal{X}(M) \tag{9}$$

holds. In fact, Hertling noticed – see Theorem 2.15 of [5] – that if $(M, \bullet, \tilde{g})$ satisfies the first of the three conditions mentioned above and "e" is the identity vector field of the multiplication "$\bullet$", then relation (9) together with the closeness of the coidentity $\tilde{g}(e)$ is equivalent to the total symmetry of the tensor $\nabla^{\tilde{g}}(\bullet)$.

Remark 2. The definition of (weak) $\mathcal{F}$-manifolds and all the properties proved about these manifolds in [1] assumed that the Euler vector field rescaled the metric and the multiplication by constants. From now on, when we refer to (weak) $\mathcal{F}$-manifolds we allow the Euler vector field to rescale the metric and the multiplication by not necessarily constant functions. In Section 3 we extend the results of [1] to this more general class of weak $\mathcal{F}$-manifolds and we prove that in fact, if the weak $\mathcal{F}$-manifold is connected and of dimension bigger than two, the Euler vector field necessarily rescales its multiplication by a constant. In Section 4 we apply our theory to the metrics obtained by non-constant conformal rescalings of the flat metrics of a Frobenius manifold.

2. A modified Saito construction

The motivation for considering such non-constant conformal rescalings comes from the following theorem.

Theorem 3. *Let $(g, \tilde{g})$ be the flat pencil of the Saito construction on the space of orbits $\mathbb{C}^n/W$ of a Coxeter group W. There is a metric $\tilde{h}$ with the following properties*:

(1) *The metric $\tilde{h}$ is flat.*
(2) *The metric $\tilde{h}$ is conformally related to the metric $\tilde{g}$: $\tilde{h} = \Omega^2\tilde{g}$, for a smooth non-vanishing function Ω.*
(3) *The metric $h := \Omega^2 g$ has constant non-zero sectional curvature s. If $s > 0$ then $\tilde{h}$ is defined on $\mathbb{C}^n/W$. If $s < 0$ then $\tilde{h}$ is defined on $\mathbb{H}^n \otimes \mathbb{C}/W$.*

Proof. We begin with a review of the salient features of the Saito construction. Details can be found in [2]. Recall that a Coxeter group of a real n-dimensional vector space $V = \mathbb{R}^n$ is a finite group of linear transformations of V generated by reflections. Let $\{t^i\}$ be a basis of W-invariant polynomials on V with degrees $\deg(t^i) = d_i$, ordered so that

$$h = d_1 > d_2 \geq \cdots \geq d_{n-1} > d_n = 2,$$

where h is the Coxeter number of the group. The action of W extends to the complexified space $V \otimes \mathbb{C} = \mathbb{C}^n$. In the Saito construction of interest is the orbit space

$$M = \mathbb{C}^n/W.$$

Starting with a W-invariant metric

$$\mathfrak{g} := \sum_{i=1}^{n} (dx^i)^2$$

on V one obtains a flat metric g on the orbit space $M\backslash\mathrm{Discr}(W)$, where $\mathrm{Discr}(W)$ is the discriminant locus of irregular orbits. What Saito showed was that there is another metric

$$\tilde{g}^* := \mathrm{Lie}_e(g^*)$$

defined on the whole of M which is also flat. Here e is the vector field which, in terms of the basis $\{t^i\}$ of invariant polynomials, is $\frac{\partial}{\partial t^1}$. The basis $\{t^i\}$ of invariant polynomials can be chosen such that the metric $\tilde{g}$ is anti-diagonal with constant entries:

$$\tilde{g}_{ij} = \delta_{i+j,n+1}$$

and is referred in this case as a Saito's basis of invariant polynomials. The key fact of the Saito construction is that the two metrics $(g, \tilde{g})$ are the regular flat pencil of a Frobenius structure on M, with Euler vector field

$$E = d_1 t^1 \frac{\partial}{\partial t^1} + \cdots + d_n t^n \frac{\partial}{\partial t^n} \tag{10}$$

and identity vector field $e = \frac{\partial}{\partial t^1}$. Suppose now that one repeats the Saito construction starting with a metric of constant sectional curvature, i.e. let

$$\mathfrak{h} := \frac{1}{\left\{c\sum_{i=1}^n (x^i)^2 + d\right\}^2}\mathfrak{g} = \frac{1}{\left\{c\sum_{i=1}^n (x^i)^2 + d\right\}^2}\sum_{i=1}^n (dx^i)^2$$

and define h to be the induced metric on (a maximal open subset of) $M\backslash\mathrm{Discr}(W)$. The metrics $\mathfrak{h}$ and h have constant sectional curvature $4(cd)$. Since one can take, without loss of generality, the invariant t^n to be

$$t^n = \sum_{i=1}^n (x^i)^2,$$

the conformal factor is a function of t^n alone. Hence one obtains a new metric

$$\tilde{h}^* := \mathrm{Lie}_e(h^*) = (ct^n + d)^2\,\mathrm{Lie}_e(g^*) = (ct^n + d)^2 \tilde{g}^*$$

defined on $\mathbb{H}^n \otimes \mathbb{C}/W$ for $(cd) < 0$ and on $\mathbb{C}^n/W$ for $(cd) > 0$. In terms of the flat coordinates $\{t^i\}$ for the metric $\tilde{g}$:

$$\tilde{h}_{ij} = \frac{1}{(ct^n + d)^2}\delta_{i+j,n+1}.$$

It remains to show that $\tilde{h}$ is flat. This may be proved using the standard formulae for transformation of the curvature tensor under a conformal change. Moreover, the flat coordinates $\{\tilde{t}^i\}$ for $\tilde{h}$ can be written down explicitly

$$\tilde{t}^1 = t^1 + \frac{c}{2(ct^n + d)}\sum_{i=2}^{n-1} t^i t^{n+1-i}, \qquad \tilde{t}^i = \frac{t^i}{ct^n + d}, \quad i = 2, \ldots, n-1,$$

$$\tilde{t}^n = \frac{at^n + b}{ct^n + d}, \quad ad - bc = 1$$

(note that this $\mathrm{SL}(2, \mathbb{C})$-transformation appears also in [2] in a slightly different context) giving

$$\tilde{h} = \sum_{i=1}^n d\tilde{t}^i d\tilde{t}^{n+1-i}.$$

The conclusion follows. □

The construction turns out to be more general.

Proposition 4. *Suppose one has a Frobenius manifold with metrics* $\tilde{g}$ *and* g (*or* η *and* g, *respectively, in Dubrovin's notation*). *Consider the conformally scaled metrics*

$$\tilde{h} = \Psi^2(t_1)\tilde{g}, \qquad h = \Psi^2(t_1)g.$$

(*Here* $(t^1, \dots, t^n)$ *are* $\tilde{g}$*-flat coordinates with* $\frac{\partial}{\partial t^1}$ *being the identity vector field* e *and* t_1 *is dual to the identity coordinate* t^1, *i.e.* $t_1 = \tilde{g}_{1i}t^i$.) *Suppose that* $\tilde{h}$ *is flat. Then* h *has constant sectional curvature.*

Proof. The curvature conditions on $\tilde{h}$ translate to a simple differential equation for the conformal factor. Solving this gives $\Psi^{-1}(t_1) = ct_1 + d$ for constants c and d. This then fixes the metric h. It can be checked (using again the standard formulae for change in the curvature tensor under a conformal change and various properties of the Christoffel symbols of g in [2]) that h has constant sectional curvature. □

Note that the conformal factor $\Omega(t^1, \dots, t^n) := \Psi(t_1)$ of Proposition 4 satisfies the condition $d\Omega \wedge g(E) = 0$, where E is the Euler vector field of the Frobenius manifold. This follows from the following easy computation: $g(E) = \tilde{g}(e) = \tilde{g}_{1i}dt^i = dt_1$, the functions $\tilde{g}_{1i}$ being constant. It turns out, as Section 4 will show, that conformally scaled metrics with this condition have particularly attractive properties.

3. $\mathcal{F}$-manifolds and compatible pencils of metrics

In this section we study the geometry of a pair of compatible metrics together with a vector field satisfying conditions (7), when D and d are not necessarily constant.

Proposition 5. *Let* $(h, \tilde{h})$ *be a regular pair of compatible metrics together with a vector field* E *on a connected manifold* M *of dimension bigger than two. Suppose that* $L_E(h) = Dh$, $L_E(\tilde{h}) = \tilde{D}\tilde{h}$, *for* $D, \tilde{D} \in C^\infty(M)$. *Then* E *rescales the multiplication "$\bullet$" associated to the pair* $(h, \tilde{h})$ *and vector field* E *if and only if* $\tilde{D} - D$ *is constant. In this case* $L_E(\bullet) = (\tilde{D} - D)\bullet$ *on* TM.

Proof. Recall that if g is an arbitrary pseudo-Riemannian metric on a manifold M and Z is a conformal vector field with $L_Z(g) = pg$ for a function $p \in C^\infty(M)$, then

$$L_Z(\nabla^g)_X(\alpha) = \tfrac{1}{2}[-dp(X)\alpha - \alpha(X)dp + g^*(\alpha, dp)g(X)]$$

for every $\alpha \in \mathcal{E}^1(M)$ and $X \in \mathcal{X}(M)$. Applying this formula to the metrics h and $\tilde{h}$ we easily get

$$L_E(\circ)(u, v) = \tfrac{1}{2}[h^*(u, d\lambda)v + h^*(u, v)d\lambda + h^*(v, dD)u - \tilde{h}^*(v, d\tilde{D})\tilde{h}h^*(u)] - Du \circ v,$$

where "$\circ$" is the multiplication (1) determined by the pair of metrics $(h, \tilde{h})$, $u, v \in \mathcal{E}^1(M)$ and $\lambda := \tilde{D} - D$. Let T be the automorphism of T^*M defined by the formula $T(u) = h(E) \circ u$. From the above relation we deduce that

$$L_E(T)(u) = \tfrac{1}{2}[E(\lambda)u + u(E)d\lambda + h^*(u, dD)h(E) - \tilde{h}^*(u, d\tilde{D})\tilde{h}(E)].$$

Recall now that the multiplications "$\circ$" and "$\bullet$" on T^*M are related by the formula $u \bullet T(v) = u \circ v$. Taking the derivative with respect to E of this formula we easily see that

$$L_E(\bullet)(u, Tv) + \tfrac{1}{2}u \bullet [E(\lambda)v + v(E)d\lambda + h^*(v, dD)h(E) - \tilde{h}^*(v, d\tilde{D})\tilde{h}(E)]$$
$$= \frac{1}{2}[h^*(u, d\lambda)v + h^*(u, v)d\lambda + h^*(v, dD)u - \tilde{h}^*(v, d\tilde{D})\tilde{h}h^*(u)] - Du \bullet T(v)$$

for every $u, v \in \mathcal{E}^1(M)$. Since $h^*\tilde{h} = E\bullet$ (the metrics $(h, \tilde{h})$ being compatible), an easy argument shows that $u \bullet \tilde{h}(E) = \tilde{h}h^*(u)$. The compatibility of $(h, \tilde{h})$ also implies, as mentioned in Section 1.1, that $h(E)$ is the identity of the multiplication "$\bullet$" on T^*M. It follows that

$$L_E(\bullet)(u, T(v)) = \tfrac{1}{2}[h^*(u, d\lambda)v + h^*(u, v)d\lambda] - \tfrac{1}{2}u \bullet [E(\lambda)v + v(E)d\lambda] - Du \bullet T(v)$$

for every $u, v \in \mathcal{E}^1(M)$. From this relation we see that E rescales the multiplication "$\bullet$" if and only if for every $u, v \in \mathcal{E}^1(M)$, the equality

$$u \bullet [E(\lambda)v + v(E)d\lambda] = h^*(u, d\lambda)v + h^*(u, v)d\lambda \tag{11}$$

holds and in this case $L_E(\bullet) = -D\bullet$ on T^*M, or $L_E(\bullet) = \lambda\bullet$ on TM. We will show that relation (11) holds only if λ is constant. Indeed, if in relation (11) we take u and v annihilating E, then we get

$$h^*(u, d\lambda)v = h^*(v, d\lambda)u$$

which can hold only if $d\lambda = \mu h(E)$ for a function $\mu \in C^\infty(M)$, since the dimension of M is at least three (and hence the annihilator of E in T^*M is of dimension at least two). Relation (11) then becomes

$$\mu[h(E, E)v + v(E)h(E)] \bullet u = \mu[u(E)v + h^*(u, v)h(E)]$$

which in turn implies that $\mu[v(E)u - u(E)v]$ is symmetric in u and v, for every $u, v \in \mathcal{E}^1(M)$. This can happen only when μ is identically zero or $\lambda = \tilde{D} - D$ is constant (M being connected). □

Remark 6. Note that relation (11) does not imply that λ is constant in dimension two. An easy argument shows that relation (11) imposes that in two dimensions the multiplication "$\bullet$" is of the form:

$$d\lambda \bullet d\lambda = 0, \qquad d\lambda \bullet h(E) = d\lambda, \qquad h(E) \bullet h(E) = h(E)$$

when λ is non-constant. Proposition 5 does not hold in dimension two: consider for example the inverse metrics

$$\tilde{h}^* = f\left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y}\right),$$

$$h^* = x\left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x}\right) + y\frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y}$$

together with the vector field

$$E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

with f smooth, non-vanishing, depending only on x, such that $\frac{xf'(x)}{f(x)}$ is non-constant on a connected open subset M of

$$\left\{(x, y) \in \mathbb{R}^2 : \frac{xf'(x)}{f(x)} \neq \frac{1}{2}, x \neq 0\right\}.$$

The pair of metrics $(h, \tilde{h})$ and the vector field E on M satisfy the hypothesis of Proposition 5, with

$$L_E(h) = h, \qquad L_E(\tilde{h}) = \left(2 - \frac{xf'(x)}{f(x)}\right)\tilde{h}.$$

Let "$\bullet$" be the multiplication associated to the pair of metrics $(h, \tilde{h})$ and vector field E. It is easy to check that $L_E(\bullet) = \left(1 - \frac{xf'(x)}{f(x)}\right) \bullet$ on TM but, as we mentioned before, $\frac{xf'(x)}{f(x)}$ is not constant. Note also that the multiplication "$\bullet$" on T^*M has the expression:

$$dx \bullet dx = 0, \qquad dx \bullet dy = dx, \qquad dy \bullet dy = dy.$$

The following definition generalizes Definition 14 of [1].

Definition 7. Let $(h, \tilde{h})$ be a compatible pair of metrics and E a vector field on a manifold M. The pair $(h, \tilde{h})$ is a weak quasi-homogeneous pencil with Euler vector field E if the following conditions are satisfied:

(1) $L_E(\tilde{h}) = \tilde{D}\tilde{h}$; $\nabla^h(E) = \frac{D}{2}\,\mathrm{Id}$, where "Id" is the identity endomorphism of TM and $D, \tilde{D} \in C^\infty(M)$.
(2) The difference $\tilde{D} - D$ is constant.

The correspondence between weak quasi-homogeneous pencils of metrics and weak $\mathcal{F}$-manifolds can be stated as in the following theorem.

Theorem 8.

(1) *Let $(h, \tilde{h})$ be a regular weak quasi-homogeneous pencil of metrics with Euler vector field E on a manifold M. Let "$\bullet$" be the multiplication on TM associated to the pair of metrics $(h, \tilde{h})$ and vector field E. Then $(M, \bullet, \tilde{h}, E)$ is a weak $\mathcal{F}$-manifold.*
(2) *Conversely, let $(M, \bullet, \tilde{h}, E)$ be a connected weak $\mathcal{F}$-manifold of dimension bigger than two. Define the metric h on M by the formula $h^*\tilde{h} = E\bullet$. The following statements hold:*
   (a) *For every $u, v \in T^*M$, $u \circ v = u \bullet T(v)$, where "$\circ$" is the multiplication (1) associated to the pair of metrics $(h, \tilde{h})$, T is the endomorphism of T^*M defined by $T(v) := h(E) \circ v$ and the multiplication "$\bullet$" on T^*M is induced by the multiplication "$\bullet$" on TM by identifying TM with T^*M using the metric $\tilde{h}$.*
   (b) *The pair $(h, \tilde{h})$ together with the vector field E is weak quasi-homogeneous. In particular, the Euler vector field E rescales the multiplication "$\bullet$" of the weak $\mathcal{F}$-manifold $(M, \bullet, \tilde{h}, E)$ by a constant.*

Proof. See the proofs of Theorem 17 and Theorem 20 of [1]—all the statements of our theorem can be proved using similar arguments, except the fact that the metrics $(h, \tilde{h})$ associated to a connected weak $\mathcal{F}$-manifold $(M, \bullet, \tilde{h}, E)$ of dimension bigger than two, together with the vector field E, satisfy the second condition of Definition 7. In order to prove this we note that, since $u \circ v = u \bullet T(v)$ for every $u, v \in T^*M$, E rescales $h, \tilde{h}$ and "$\bullet$", $u \bullet \tilde{h}(E) = \tilde{h}h^*(u)$ (easy check) and $h(E)$ is the identity of the multiplication "$\bullet$" on T^*M (trivial consequence of how h was defined), E must necessarily rescale the multiplication "$\bullet$" on TM by a constant (just repeat the proof of Proposition 5). Since $h^*\tilde{h} = E\bullet$, the second condition of Definition 7, applied to $(h, \tilde{h})$ and E, holds. □

The following theorem and its corollary generalize the results from Section 6 of [1].

Theorem 9. *Let $(M, \bullet, \tilde{h}, E)$ be a weak $\mathcal{F}$-manifold with $L_E(\tilde{h}) = \tilde{D}\tilde{h}$, $L_E(\bullet) = k\bullet$, where k is constant and $\tilde{D} \in C^\infty(M)$. Let h be the metric on M defined by the relation $h^*\tilde{h} = E\bullet$. Consider the multiplication "$\bullet$" also on T^*M, by identifying TM and T^*M using the metric $\tilde{h}$.*

Then $(M, \bullet, \tilde{h}, E)$ *is an* $\mathcal{F}$*-manifold if and only if the equality*

$$R^h_{X,Y}(\alpha) = R^{\tilde{h}}_{X,Y}(\alpha) + (-R^{\tilde{h}}_{X,E}(\alpha) + \tfrac{1}{2}\alpha(X)d\tilde{D}) \bullet \tilde{h}(E^{-1} \bullet Y) - (-R^{\tilde{h}}_{Y,E}(\alpha) + \tfrac{1}{2}\alpha(Y)d\tilde{D}) \bullet \tilde{h}(E^{-1} \bullet X)$$

holds, for every $X, Y \in \mathcal{X}(M)$ *and* $\alpha \in \mathcal{E}^1(M)$.

Proof. The argument is similar to the one employed in the proof of Theorem 23 of [1]. The only difference from the case studied in [1] is that $\tilde{D}$ can be non-constant and then

$$\nabla^{\tilde{h}}_Y(\nabla^{\tilde{h}} E)_X = R^{\tilde{h}}_{Y,E}(X) + \tfrac{1}{2}[-\tilde{h}(X, Y)d(\tilde{D}) + Y(\tilde{D})\tilde{h}(X) + X(\tilde{D})\tilde{h}(Y)]$$

for every $X, Y \in \mathcal{X}(M)$, which is the analogue of Lemma 22 of [1] and can be proved in the same way in this more general context. □

Corollary 10. *Consider the set-up of Theorem 9 and suppose that* $\tilde{h}$ *is flat. Then* $(M, \bullet, \tilde{h}, E)$ *is an* $\mathcal{F}$*-manifold if and only if* h *has constant sectional curvature* s *and* $d\tilde{D} = -2sh(E)$.

Proof. From Theorem 9 and the flatness of $\tilde{h}$ we know that $(M, \bullet, \tilde{h}, E)$ is an $\mathcal{F}$-manifold if and only if the curvature R^h of h has the following expression:

$$R^h_{X,Y}(\alpha) = \tfrac{1}{2}d\tilde{D} \bullet (\alpha(X)\tilde{h}(E^{-1} \bullet Y) - \alpha(Y)\tilde{h}(E^{-1} \bullet X)) \quad (12)$$

for every $X, Y \in \mathcal{X}(M)$ and $\alpha \in \mathcal{E}^1(M)$. It is clear now that if h has constant sectional curvature s and $d\tilde{D} = -2sh(E)$, then relation (12) is satisfied and hence $(M, \bullet, \tilde{h}, E)$ is an $\mathcal{F}$-manifold. Conversely, suppose that $(M, \bullet, \tilde{h}, E)$ is an $\mathcal{F}$-manifold, so that relation (12) is satisfied. Then

$$h(R^h_{X,Y}Z, V) = \tfrac{1}{2}(h(X, Z)d\tilde{D}(V \bullet Y \bullet E^{-1}) - h(Y, Z)d\tilde{D}(V \bullet X \bullet E^{-1})) \quad (13)$$

for every $X, Y, Z, V \in \mathcal{X}(M)$. On the other hand, since

$$h(R^h_{X,Y}Y, X) = -h(R^h_{X,Y}X, Y) \quad \forall X, Y \in \mathcal{X}(M)$$

we easily get

$$h(X, X)(d\tilde{D})(Y^2 \bullet E^{-1}) = h(Y, Y)(d\tilde{D})(X^2 \bullet E^{-1})$$

or

$$h(X, T)(d\tilde{D})(Y \bullet S \bullet E^{-1}) = h(Y, S)(d\tilde{D})(X \bullet T \bullet E^{-1}) \quad \forall X, Y, S, T \in \mathcal{X}(M).$$

It follows that $h(E) \wedge d\tilde{D} = 0$ (let $S = T := E$ in the above relation) or $d\tilde{D} = -2sh(E)$, for a function $s \in C^\infty(M)$. From relation (13) we deduce that s is constant and h has constant sectional curvature s. □

4. The geometry of conformally scaled compatible pencils

In this section we fix a pair of metrics $(g, \tilde{g})$ on a manifold M. The following lemma will be relevant in our calculations.

Lemma 11. *Suppose that the metrics* $(g, \tilde{g})$ *are almost compatible. Then, for every* $X, Y \in \mathcal{X}(M)$ *and* $\alpha \in \mathcal{E}^1(M)$ *the relation*

$$g^*(\nabla^{\tilde{g}}_X\alpha - \nabla^g_X\alpha, \tilde{g}(Y)) = g^*(\nabla^{\tilde{g}}_Y\alpha - \nabla^g_Y\alpha, \tilde{g}(X))$$

holds.

Proof. Let $X := \tilde{g}^*(\gamma)$ and $Y := \tilde{g}^*(\delta)$, for $\gamma, \delta \in \mathcal{E}^1(M)$. Then

$$g^*(\nabla^{\tilde{g}}_X \alpha - \nabla^g_X \alpha, \tilde{g}(Y)) = \delta(g^*(\nabla^{\tilde{g}}_{\tilde{g}^*(\gamma)} \alpha - \nabla^g_{\tilde{g}^*(\gamma)} \alpha)) = \delta(\tilde{g}^*(\nabla^{\tilde{g}}_{g^*(\gamma)} \alpha - \nabla^g_{g^*(\gamma)} \alpha))$$
$$= \tilde{g}^*(\gamma \circ \alpha, \delta) = \tilde{g}^*(\gamma, \delta \circ \alpha) = g^*(\nabla^{\tilde{g}}_Y \alpha - \nabla^g_Y \alpha, \tilde{g}(X)),$$

where "$\circ$" is the multiplication (1) associated to the pair $(h, \tilde{h})$ and we have used relations (3) and (4). □

As a consequence of Lemma 11 we deduce that the compatibility property of two metrics is conformal invariant.

Proposition 12. *Suppose that the metrics $(g, \tilde{g})$ are compatible and let $\Omega \in C^\infty(M)$, non-vanishing. Then the metrics $(h := \Omega^2 g, \tilde{h} := \Omega^2 \tilde{g})$ are also compatible.*

Proof. It is obvious that the metrics h and $\tilde{h}$ are almost compatible, since $h^* \tilde{h} = g^* \tilde{g}$ (and hence the integrability tensor of $h^* \tilde{h}$, being equal to that of $g^* \tilde{g}$, is identically zero). In order to show the compatibility of $(h, \tilde{h})$, we first notice that

$$\nabla^h_X \alpha = \nabla^g_X \alpha - \frac{d\Omega}{\Omega}(X)\alpha - \alpha(X)\frac{d\Omega}{\Omega} + g^*\left(\alpha, \frac{d\Omega}{\Omega}\right) g(X),$$
$$\nabla^{\tilde{h}}_X \alpha = \nabla^{\tilde{g}}_X \alpha - \frac{d\Omega}{\Omega}(X)\alpha - \alpha(X)\frac{d\Omega}{\Omega} + \tilde{g}^*\left(\alpha, \frac{d\Omega}{\Omega}\right) \tilde{g}(X)$$

for every $X \in \mathcal{X}(M)$ and $\alpha \in \mathcal{E}^1(M)$, from where we deduce that

$$\nabla^{\tilde{h}}_X \alpha - \nabla^h_X \alpha = \nabla^{\tilde{g}}_X \alpha - \nabla^g_X \alpha + \tilde{g}^*\left(\alpha, \frac{d\Omega}{\Omega}\right) \tilde{g}(X) - g^*\left(\alpha, \frac{d\Omega}{\Omega}\right) g(X). \tag{14}$$

To prove the compatibility of the metrics $(h, \tilde{h})$ we shall verify relation (5). Notice that, since $h^* = \Omega^{-2} g^*$, we need to show that the relation

$$g^*(\nabla^{\tilde{h}}_X \alpha - \nabla^h_X \alpha, \nabla^{\tilde{h}}_Y \beta - \nabla^h_Y \beta) = g^*(\nabla^{\tilde{h}}_Y \alpha - \nabla^h_Y \alpha, \nabla^{\tilde{h}}_X \beta - \nabla^h_X \beta) \tag{15}$$

holds, for every $X, Y \in \mathcal{X}(M)$ and $\alpha, \beta \in \mathcal{E}^1(M)$. Using the compatibility of the metrics $(g, \tilde{g})$ and relation (14), we easily see that relation (15) is equivalent with

$$\tilde{g}^*\left(\beta, \frac{d\Omega}{\Omega}\right) [g^*(\nabla^{\tilde{g}}_X \alpha - \nabla^g_X \alpha, \tilde{g}(Y)) - g^*(\nabla^{\tilde{g}}_Y \alpha - \nabla^g_Y \alpha, \tilde{g}(X))]$$
$$+ \tilde{g}^*\left(\alpha, \frac{d\Omega}{\Omega}\right) [g^*(\nabla^{\tilde{g}}_Y \beta - \nabla^g_Y \beta, \tilde{g}(X)) - g^*(\nabla^{\tilde{g}}_X \beta - \nabla^g_X \beta, \tilde{g}(Y))] = 0,$$

which is obviously true from Lemma 11. □

For the rest of this section we suppose that the metrics $(g, \tilde{g})$ are the regular flat metrics of a Frobenius manifold $(M, \bullet, \tilde{g}, E)$. We study the geometry of the pair of scaled metrics $(h := \Omega^2 g, \tilde{h} := \Omega^2 \tilde{g})$ together with the vector field E. We restrict to the case when the scaled pair is regular and we denote by "$\bullet_h$" the associated multiplication on TM or T^*M. Recall that the multiplication "$\bullet_g$" on TM associated to the pair of metrics $(g, \tilde{g})$ together with E coincides with the multiplication "$\bullet$" of the Frobenius manifold $(M, \bullet, \tilde{g}, E)$.

Proposition 13. *The multiplications "$\bullet_h$" and "$\bullet_g$" coincide on TM.*

Proof. From relation (14) we easily see that the multiplications "$\circ_h$" and "$\circ_g$" associated to the pair of metrics $(h, \tilde{h})$ and $(g, \tilde{g})$, respectively, are related by the formula

$$\alpha \circ_h \beta = \Omega^{-2}\left[\alpha \circ_g \beta + g^*\left(\beta, \frac{d\Omega}{\Omega}\right)\alpha - \tilde{g}^*\left(\beta, \frac{d\Omega}{\Omega}\right)\tilde{g}g^*(\alpha)\right] \tag{16}$$

for every $\alpha, \beta \in \mathcal{E}^1(M)$. Define the automorphisms $T(\alpha) := g(E) \circ_g \alpha$ and $\tilde{T}(\alpha) := h(E) \circ_h \alpha$ of T^*M. Relation (14) also implies that

$$\tilde{T}(\alpha) = T(\alpha) + g^*\left(\alpha, \frac{d\Omega}{\Omega}\right) g(E) - \tilde{g}^*\left(\alpha, \frac{d\Omega}{\Omega}\right)\tilde{g}(E). \tag{17}$$

Since $\alpha \circ_h \beta = \alpha \bullet_h \tilde{T}(\beta)$ and similarly $\alpha \circ_g \beta = \alpha \bullet_g T(\beta)$ we deduce from (16) and (17) that the relation

$$\begin{aligned}\alpha \bullet_h &\left[T(\beta) + g^*\left(\beta, \frac{d\Omega}{\Omega}\right) g(E) - \tilde{g}^*\left(\beta, \frac{d\Omega}{\Omega}\right)\tilde{g}(E)\right]\\ &= \Omega^{-2}\left[\alpha \bullet_g T(\beta) + g^*\left(\beta, \frac{d\Omega}{\Omega}\right)\alpha - \tilde{g}^*\left(\beta, \frac{d\Omega}{\Omega}\right)\tilde{g}g^*(\alpha)\right]\end{aligned}$$

holds, for every $\alpha, \beta \in \mathcal{E}^1(M)$. As in the proof of Proposition 5, $\alpha \bullet_h \tilde{g}(E) = \Omega^{-2}\tilde{h}h^*(\alpha)$ and $\alpha \bullet_h g(E) = \Omega^{-2}\alpha$ (the metrics $(h, \tilde{h})$ being compatible). It follows that $\bullet_h = \Omega^{-2}\bullet_g$ on T^*M, or $\bullet_h = \bullet_g$ on TM. □

Proposition 14. *The following statements are equivalent*:

(1) *the pair $(h, \tilde{h})$ is weak quasi-homogeneous with Euler vector field E*;
(2) $g(E) \wedge d\Omega = 0$;
(3) $(M, \bullet, \tilde{h}, E)$ *is an $\mathcal{F}$-manifold*;
(4) $(M, \bullet, \tilde{h}, E)$ is a weak $\mathcal{F}$-manifold.

Proof. Before proving the equivalence of the statements, we make some preliminary remarks. Since $(g, \tilde{g})$ are the flat metrics of a Frobenius manifold, $L_E(g) = (1 - d)g$ and $L_E(\tilde{g}) = D\tilde{g}$ for some constants D and d. It follows that

$$L_E(h) = \left(1 - d + \frac{2E(\Omega)}{\Omega}\right)h, \qquad L_E(\tilde{h}) = \left(D + \frac{2E(\Omega)}{\Omega}\right)\tilde{h}.$$

Also,

$$\begin{aligned}\nabla^h_X(E) &= \nabla^g_X(E) + \frac{d\Omega}{\Omega}(X)E + \frac{E(\Omega)}{\Omega}X - g(X, E)g^*\left(\frac{d\Omega}{\Omega}\right)\\ &= \left(\frac{1-d}{2} + \frac{E(\Omega)}{\Omega}\right)X - \left(E \wedge g^*\left(\frac{d\Omega}{\Omega}\right)\right)(g(X)).\end{aligned}$$

Moreover, from Proposition 12 we know that the metrics $(h, \tilde{h})$ are compatible. The equivalence $1 \Longleftrightarrow 2$ clearly follows from these facts. The equivalence $2 \Longleftrightarrow 3$ follows from Hertling's observation mentioned at the end of Section 1.1: indeed, condition (2) means that the coidentity $\tilde{h}(e) = \Omega^2 g(E)$ is closed (note that the 1-form $g(E)$, being equal to $\tilde{g}(e)$, is closed because e is $\nabla^{\tilde{g}}$-parallel). To prove the equivalence $3 \Longleftrightarrow 4$ we notice that, since $(M, \bullet, \tilde{g}, E)$ is an $\mathcal{F}$-manifold, the (3, 1)-tensor field $\nabla^{\tilde{h}}(\bullet)$ (corresponding to the (4, 0)-tensor field $\nabla^{\tilde{h}}(\bullet)$ using the metric $\tilde{h}$) satisfies the relation

$$\nabla^{\tilde{h}}_X(\bullet)(Y, Z) - \nabla^{\tilde{h}}_Y(\bullet)(X, Z) = \left(\tilde{g}^*\left(\frac{d\Omega}{\Omega}\right) \wedge e\right)(\tilde{g}(Y \bullet Z)) \bullet X - \left(\tilde{g}^*\left(\frac{d\Omega}{\Omega}\right) \wedge e\right)(\tilde{g}(X \bullet Z)) \bullet Y$$

for every $X, Y, Z \in \mathcal{X}(M)$. Suppose now that $(M, \bullet, \tilde{h}, E)$ is a weak $\mathcal{F}$-manifold. The symmetry

$$\nabla^{\tilde{h}}_E(\bullet)(Y, Z) = \nabla^{\tilde{h}}_Y(\bullet)(E, Z) \quad \forall Y, Z \in \mathcal{X}(M)$$

of the (3, 1)-tensor field $\nabla^{\tilde{h}}(\bullet)$ becomes, after replacing Z with $E^{-1} \bullet Z$, the relation:

$$\left(\tilde{g}^*\left(\frac{d\Omega}{\Omega}\right) \wedge e\right)(\tilde{g}(Z)) \bullet Y = \left(\tilde{g}^*\left(\frac{d\Omega}{\Omega}\right) \wedge e\right)(\tilde{g}(Y \bullet Z \bullet E^{-1})) \bullet E.$$

It is clear now that if $(M, \bullet, \tilde{h}, E)$ is a weak $\mathcal{F}$-manifold, then

$$\nabla^{\tilde{h}}_X(\bullet)(Y, Z) = \nabla^{\tilde{h}}_Y(\bullet)(X, Z) \quad \forall\, X, Y, Z \in \mathcal{X}(M)$$

which implies that $(M, \bullet, \tilde{h}, E)$ is an $\mathcal{F}$-manifold (the symmetry of the (4, 0)-tensor field $\nabla^{\tilde{h}}(\bullet)$ in the last three arguments is a consequence of the fact that $\tilde{h}$ is "$\bullet$"-invariant and of the commutativity of "$\bullet$"). The equivalence $3 \iff 4$ follows. □

Note that Proposition 14 together with Corollary 10 provide a different view-point of Proposition 4.

Corollary 15. *Let $(g, \tilde{g})$ be the flat metrics of a Frobenius manifold $(M, \bullet, \tilde{g}, E)$. Let $\Omega \in C^\infty(M)$ non-vanishing which satisfies $d\Omega \wedge g(E) = 0$. Consider the scaled metrics $(h := \Omega^2 g, \tilde{h} := \Omega^2 \tilde{g})$. If $\tilde{h}$ is flat, then h has constant sectional curvature.*

Proof. The condition $d\Omega \wedge g(E) = 0$ implies, using Proposition 14, that $(M, \bullet, \tilde{h}, E)$ is an $\mathcal{F}$-manifold. The conclusion follows from Corollary 10 (since $h^*\tilde{h} = E\bullet$ and $\tilde{h}$ is flat). □

5. The modified Saito construction revisited

We return now to the modified Saito construction described in Section 2, summarizing the various results in the following theorem.

Theorem 16. *Let $(g, \tilde{g})$ be the flat metrics of the Frobenius structure $(M, \bullet, \tilde{g}, E)$ on the space of orbits $M = \mathbb{C}^n / W$ of a Coexter group W. Let $\{t^i\}$ be a Saito basis of W-invariant polynomials with $\deg(t^i) = d_i$, in which the metric $\tilde{g}$ is anti-diagonal, the identity vector field e is $\frac{\partial}{\partial t^1}$ and the Euler vector field E has the expression (10). Consider the pair of scaled metrics $(h := \Omega^2 g, \tilde{h} := \Omega^2 \tilde{g})$, where $\Omega \in C^\infty(M_0)$ is non-vanishing on an open subset M_0 of M. The following facts hold*:

(1) *The metrics $(h, \tilde{h})$ are compatible on M_0.*
(2) *The metrics $(h, \tilde{h})$ together with the Euler vector field E is a weak quasi-homogeneous pencil on M_0 if and only if Ω depends only on the last coordinate t^n. If $\Omega = \Omega(t^n)$ and the weak quasi-homogeneous pair $(h, \tilde{h})$ is also regular, then the associated weak $\mathcal{F}$-manifold is $(M_0, \bullet, \tilde{h}, E)$ and is an $\mathcal{F}$-manifold.*

(3) *Let* $\Omega(t) = (ct^n + d)^{-1}$, *for* c, d *constants. The pair* $(h, \tilde{h})$ *together with* E *is weak quasi-homogeneous on* $\mathbb{H}^n \otimes \mathbb{C}/W$ (*when* $(cd) < 0$) *and on* $\mathbb{C}^n/W$ (*when* $(cd) > 0$). *It is regular on the open subset where*

$$t^n \neq \frac{d}{c}, \qquad t^n \neq \frac{(1-d_1)d}{(1+d_1)c}.$$

Moreover, $\tilde{h}$ *is flat and* h *has constant sectional curvature* $4(cd)$.

Proof. The proof is obvious by now, except perhaps the regularity of $(h, \tilde{h})$, when $\Omega(t) := (ct^n + d)^{-1}$. Note that the endomorphism $T(u) := h(E) \circ_h u$ has the following expression:

$$T(u) = \sum_{i=1}^{n} \left((d_i - 1)u_i + \frac{cu_1}{ct^n + d} d_{n-i+1} t^{n-i+1} \right) dt^i - \frac{cu(E)}{ct^n + d} dt^n$$

for every $u = \sum_{i=1}^{n} u_i dt^i$. The regularity condition can now be easily checked. □

Thus given a Frobenius manifold with prepotential F one may conformally rescale the metrics, derive new flat coordinates and multiplication, and calculate the new prepotential $\tilde{F}$ (though, from Proposition 14, the two multiplications $\bullet_g$ and $\bullet_h$ coincide).

$$\begin{array}{lcl} F & \to & \{\bullet_g, \tilde{g}\} \\ \downarrow & & \downarrow \\ \tilde{F} & \leftarrow & \{\bullet_h, \tilde{h}\} \end{array}$$

This gives rise to an $SL(2, \mathbb{C})$-symmetry on solution space of the WDVV equation.

Example 17. Starting with the prepotential[1]

$$F = \tfrac{1}{2}t_1^2 t_3 + \tfrac{1}{2}t_1 t_2^2 + f(t_2, t_3),$$

where f satisfies the differential equation:

$$f_{333} = f_{223}^2 - f_{233} f_{222}$$

one obtains the new solution

$$\tilde{F} = \frac{1}{2}\tilde{t}_1^2 t_3 + \frac{1}{2}\tilde{t}_1 \tilde{t}_2^2 + \left\{ \frac{c\tilde{t}_2^4}{8(c\tilde{t}_3 + d)} + (c\tilde{t}_3 + d)^2 f\left(\frac{\tilde{t}_2}{c\tilde{t}_3 + d}, \frac{a\tilde{t}_3 + b}{c\tilde{t}_3 + d} \right) \right\},$$

where $ad - bc = 1$. Note that this is a transformation on solutions on the WDVV equation: the transformation breaks the linearity condition on the Euler vector field (except in the very special case identified in [2]) and so does not generate new examples of Frobenius manifolds.

As mentioned in Section 1, these conformally flat pencils will automatically generate bi-Hamiltonian structures and hence certain integrable hierarchies of evolution equations. The properties of these hierarchies will be considered elsewhere.

[1] For notational convenience indices are dropped in these example *only*, so $t_i = t^i$.

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